The Sphere Packing Problem

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Those who heard the talk will surely remember that I played Carl Orff's "Carmina Burana" as the audience was being seated. I felt that drastic measures were needed to wake people up at 08:30.

My original plan was to begin the talk with a second piece of music. I was going to start by digressing to talk about a proposal I made 2 or 3 years ago: the "Eternal Home Page". This is a possible Internet service that some major organization such as the IEEE, Harvard University, AT&T, or even the Vatican, might offer: a home page "in perpetuity". Such a "perpetual page" or "eternity page" would be a home page that the organization would help the customer set up, with a guarantee that it would last for say 500 years, or until the organization no longer exists. The details are described on my web site¹. Almost everyone wants to be remembered by posterity, and I still think this is one of my best ideas, even though I haven't succeeded in getting any organization interested in the proposal. At this point I was planning to play a recording of Jessye Norman singing the moving and unforgettable aria "Remember Me" from Henry Purcell's "Dido and Aeneas". However, in the end I omitted this part of the talk for lack of time.

What follows is a heavily abridged version of the talk. For much more information (and references) see the Introduction² to the Third Edition of my book with John Conway "Sphere Packings, Lattices and Groups".

The original title of the talk was "Codes and Lattices", but in the end — keeping in mind my slogan that "codes are to lattices as rock and roll is to classical music" — I decided to talk mostly about sphere packings and lattices.

Although the sphere packing problem has a long history in geometry, the real story begins (of course) with Shannon. The connection is via the sampling theorem. As Shannon observes in his classic 1948 paper, if f is a signal of bandwidth W hertz, with almost all its energy concentrated in an interval of T secs, then f is accurately represented by a vector of 2WT samples, which may be regarded as the coordinates of a single point in \mathbb{R}^n , n=2WT. Nearly equal signals are represented by neighboring points, so to keep the signals distinct, Shannon represents them by n-dimensional 'billiard balls', and is therefore led to ask: what is the best way to pack 'billiard balls' in n dimensions?

Ambrose Rogers' book on "Packings and Coverings" came out in 1964, and my plan in this talk was to imagine that I was giving a report to Claude Shannon on some of the progress that has been made since 1964. This year is a natural break point in the subject, since John Leech discovered his great 24-dimensional lattice sphere packing around 1965, and this triggered many of the subsequent developments. The report proceeds upwards in dimension, beginning in 2 dimensions.

 $^{^{1}}www.research.att.com/{\sim}njas/eternal.html$

²Available from my home page www.research.att.com/~njas/

Dimension 2

The best packing in dimension 2 is the familiar 'hexagonal lattice' packing of circles, each touching six others. The centers are the points of the root lattice A_2 . The density Δ of this packing is the fraction of the plane occupied by the spheres: $\pi/\sqrt{12} = 0.9069...$

In general we wish to find Δ_n , the highest possible density of a packing of equal nonoverlapping spheres in \mathbb{R}^n , or $\Delta_n^{(L)}$, the highest density of any packing in which the centers form a lattice. It is known (Fejes Tóth, 1940) that $\Delta_2 = \Delta_2^{(L)} = \pi/\sqrt{12}$.

We are also interested in packing points on a sphere, and especially in the 'kissing number problem': find τ_n (resp. $\tau_n^{(L)}$), the maximal number of spheres that can touch an equal sphere in \mathbb{R}^n (resp. in any lattice in \mathbb{R}^n). It is trivial that $\tau_2 = \tau_2^{(L)} = 6$.

Dimension 3

Just a week before this talk, Thomas Hales at the Univ. of Michigan made a dramatic announcement: he had finally settled the 387-year old Kepler conjecture³ that no 3-dimensional packing has greater density than the face-centered cubic (or f.c.c) lattice A_3 , or in other words $\Delta_3 = \Delta_3^{(L)} = \pi/\sqrt{18}$.

In two dimensions the hexagonal lattice is (a) the densest lattice packing, (b) the least dense lattice covering, and (c) is geometrically similar to its dual lattice. There is a little-known three-dimensional lattice that is similar to its dual, and, among all lattices with this property, is both the densest packing and the least dense covering. This is the m.c.c. (or mean-centered cuboidal) lattice, which is in a sense is the geometric mean of the f.c.c. lattice and its dual the body-centered cubic (b.c.c.) lattice (Conway and the author, 1994).

Dimensions 4–8

Table 1 summarizes what is presently known about the sphere packing and kissing number problems in dimensions ≤ 24 . Entries enclosed inside a solid line are known to be optimal, those inside a dashed line optimal among lattices.

The large box in the 'density' column refers to Blichfeldt's 1935 result that the root lattices $\mathbb{Z} \simeq A_1, A_2, A_3 \simeq D_3, D_4, D_5, E_6, E_7, E_8$ achieve $\Delta_n^{(L)}$ for $n \leq 8$. It is remarkable that more than 60 years later $\Delta_n^{(L)}$ is still unknown.

The large box in the right-hand column refers to Watson's 1963 result that the kissing numbers of the above lattices, together with that of the laminated lattice Λ_9 , achieve $\tau_n^{(L)}$ for $n \leq 9$. Andrew Odlyzko and I, and independently Vladimir Levenshtein, determined τ_8 and τ_{24} . The packings achieving these two bounds are unique.

Dimension 9. Laminated lattices

There is a simple construction, the 'laminating' or 'greedy' construction, that produces many of the densest lattices in dimensions up to 26. Let Λ_1 denote the even integers in \mathbb{R}^1 , and define the *n*-dimensional laminated lattices Λ_n recursively by: consider all lattices of minimal norm 4 that contain some Λ_{n-1} as a sublattice, and select those of greatest density. It had

³See N.J.A. Sloane, "Kepler Conjecture Confirmed", Nature, Oct. 1, 1998, p. 435.

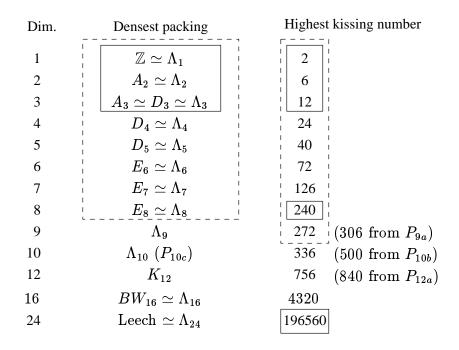


Table 1: Densest packings and highest kissing numbers known in low dimensions. (Parenthesized entries are nonlattice arrangements that are better than any known lattice.)

been known since the 1940's that this produces the densest lattices known for $n \leq 10$. In 1982 Conway and I determined *all* inequivalent laminated lattices for $n \leq 25$, and found the density of Λ_n for $n \leq 48$ (Fig. 1). A key result needed for this was the determination of the covering radius of the Leech lattice and the enumeration of the deep holes in that lattice.

What are all the best sphere packings in low dimensions? In a 1995 paper Conway and I describe what may be all the best packings in dimensions $n \leq 10$, where 'best' means both having the highest density and not permitting any local improvement. In particular, we conjecture that $\Delta_n^{(L)} = \Delta_n$ for $n \leq 9$. For example, it appears that the best five-dimensional sphere packings are parameterized by the 4-colorings of \mathbb{Z} . We also find what we believe to be the exact numbers of 'uniform' packings among these, those in which the automorphism group acts transitively. These assertions depend on certain plausible but as yet unproved postulates.

A remarkable property of 9-dimensional packings. We also showed that the laminated lattice Λ_9 has the following astonishing property. Half the spheres can be moved bodily through arbitrarily large distances without overlapping the other half, only touching them at isolated instants, the density remaining the same at every instant. All these packings have the same density, which we conjecture is the value of $\Delta_9 = \Delta_9^{(L)}$. Another result in the same paper is that there are extraordinarily many 16-dimensional packings that are just as dense as the Barnes-Wall lattice $BW_{16} \simeq \Lambda_{16}$.

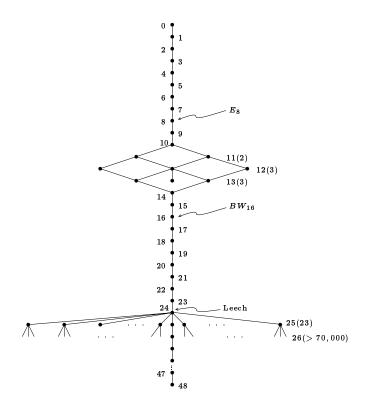


Figure 1: Inclusions among laminated lattices Λ_n .

Dimension 10. Construction A.

In dimension 10 we encounter for the first time a nonlattice packing that is denser than all known lattices. This packing, and the nonlattice packing with the highest known kissing number in dimension 9, are easily obtained from 'Construction A' (Leech and Sloane, 1969). If \mathcal{C} is a binary code of length n, the corresponding packing is $P(\mathcal{C}) = \{x \in \mathbb{Z}^n : x \pmod{2} \in \mathcal{C}\}$.

Consider the vectors $abcde \in (\mathbb{Z}/4\mathbb{Z})^5$ where $b, c, d \in \{+1, -1\}$, a = c - d, e = b + c, together with all their cyclic shifts, and apply the 'Gray map' $0 \to 00$, $1 \to 01$, $2 \to 11$, $3 \to 10$ to obtain a binary code \mathcal{C}_{10} containing 40 vectors of length 10 and minimal distance 4. This is our description of a code first discovered by Best. Then $P(\mathcal{C}_{10}) = P_{10c}$ is the record 10-dimensional packing.

Higher Dimensions

Figure 2 shows the density of the best packings known up to dimension 48, rescaled to make them easier to read. The vertical axis gives $\log_2 \delta + n(24-n)/96$, Where the center density δ is the density Δ divided by the volume of a unit sphere. Lattice packings are indicated by small circles, nonlattices by crosses (however, the locations of the lattices are only approximate). The figure is dominated by the two arcs of the graph of the laminated lattices Λ_n , which touch the zero ordinate at n=0, 24 (the Leech lattice) and 48. K_{12} is the Coxeter-Todd lattice, Q_{32} is Quebbemann's lattice, and P_{48q} is an extremal unimodular lattice constructed from a self-dual code of length 48 over GF(3).

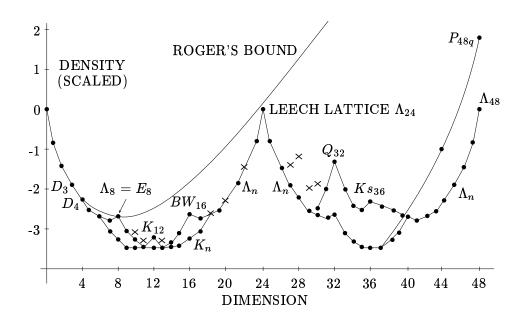


Figure 2: Densest sphere packings known in dimensions $n \leq 48$.

That's about the first half of the talk. But I've run out of space. For descriptions of the beautiful packings in Fig. 2, and much more, see the Introduction⁴ to the Third Edition of my book with John Conway "Sphere Packings, Lattices and Groups".

 $^{^4}$ Available from my home page www.research.att.com/ \sim njas/